

MCS 252 MT1 Questions

- 1) Use the transformation $x = u + \frac{v}{2}$, $y = v$ to evaluate the integral

$$\int_0^2 \int_{\frac{y}{2}}^{\frac{(y+4)}{2}} y^3(2x-y)e^{(2x-y)^2} dx dy.$$

$$R = \begin{cases} x = \frac{y}{2}, & y=0 \\ x = \frac{y+4}{2}, & y=2 \end{cases}$$

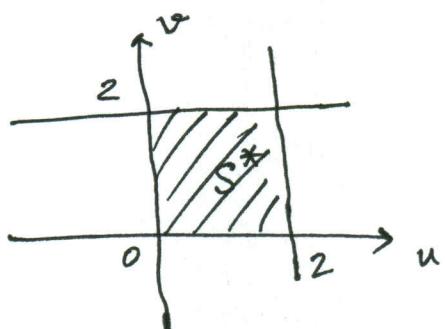
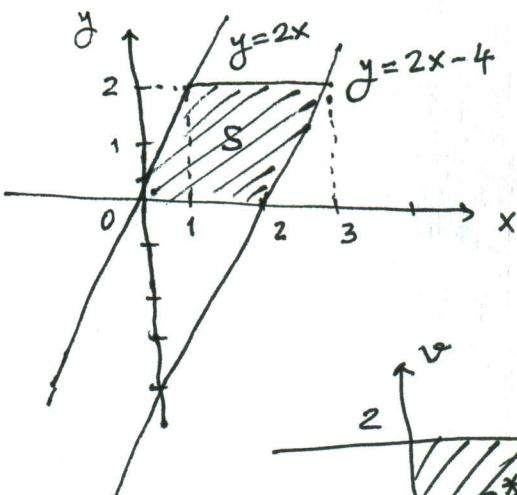
$$\text{and } T = \begin{cases} x = u + \frac{v}{2} \\ y = v \end{cases} \Rightarrow \begin{cases} u = \frac{2x-y}{2} \\ v = y \end{cases}$$

$$y=0 \Rightarrow v=0$$

$$y=2 \Rightarrow v=2$$

$$y=2x \Rightarrow u=0$$

$$y=2x-4 \Rightarrow u=2$$



$$|J| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} 1 & 1/2 \\ 0 & 1 \end{vmatrix} = \boxed{1}$$

$$\therefore = \int_0^2 \int_0^2 v^3 \cdot 2u \cdot e^{4u^2} \cdot 1 dv du$$

$$= \int_0^2 2u e^{4u^2} \cdot \left(\frac{v^4}{4} \Big|_0^2 \right) du = \int_0^2 8u e^{4u^2} du$$

$$= \int_0^{16} 8u \cdot e^t \cdot \frac{dt}{8u} = \boxed{e^{16} - 1}$$

$$\begin{aligned} t &= 4u^2 \\ dt &= 8u du \\ u=0 &\rightarrow t=0 \\ u=2 &\rightarrow t=16 \end{aligned}$$

2) Write the integral

$$\int_0^2 \int_0^{\sqrt{1-(x-1)^2}} \frac{x+y}{x^2+y^2} dy dx$$

using polar coordinates. (DO NOT EVALUATE THE INTEGRAL).

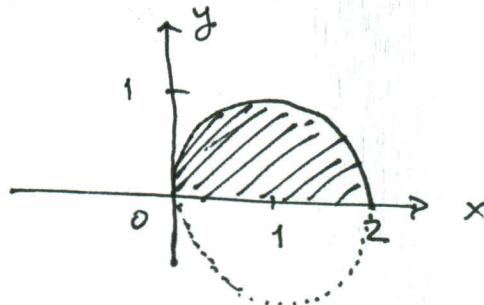
$$R = \begin{cases} y=0, \quad y=\sqrt{1-(x-1)^2} \\ x=0, \quad x=2 \end{cases} \Rightarrow (x-1)^2 + y^2 = 1$$

$$x=r\cos\theta \quad y=r\sin\theta, \quad dy dx \leftrightarrow r dr d\theta$$

$$\pi/2 \quad 2\cos\theta$$

$$= \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{r(\cos\theta + \sin\theta)}{r^2} r dr d\theta$$

$$= \int_0^{\pi/2} \left(\int_0^{2\cos\theta} (\cos\theta + \sin\theta) dr \right) d\theta$$



$$y^2 + (x-1)^2 = 1$$

$$\Rightarrow r^2 = 2r\cos\theta$$

$$r = 2\cos\theta$$

3) Let

$$T = \begin{cases} u(x, y) = x^3 - xy + 2 \\ v(x, y) = y - \frac{3x^2}{2} \end{cases}$$

be given.

a) Show that $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is locally invertible for $y \neq 0$.

b) Compute $\frac{\partial y}{\partial u}$ and $\frac{\partial x}{\partial v}$.

$$\text{a)} \quad DT = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 3x^2 - y & -x \\ -3x & 1 \end{vmatrix} = \cancel{3x^2 - y - 3x^2} \\ = -y \neq 0$$

Since $y \neq 0$, by the Inverse Function Theorem, T is locally invertible.

$$\text{b)} \quad \frac{\partial(u, v)}{\partial(x, y)} = \begin{bmatrix} 3x^2 - y & -x \\ -3x & 1 \end{bmatrix}^{-1} = \begin{pmatrix} 1 \\ -y \end{pmatrix} \cdot \text{adj} A$$

$$= \left(-\frac{1}{y} \right) \begin{pmatrix} 1 & x \\ 3x & 3x^2 - y \end{pmatrix} = \begin{bmatrix} -\frac{1}{y} & -\frac{x}{y} \\ -\frac{3x}{y} & \frac{-3x^2 + y}{y} \end{bmatrix}$$

$$\therefore \boxed{\frac{\partial y}{\partial u} = -\frac{3x}{y}} \quad , \quad \boxed{\frac{\partial x}{\partial v} = -\frac{x}{y}}$$

4) Let

$$\begin{cases} 5x - 4y + z^3 - 3u + v^2 = 0 \\ -2x + y^2 + z + u^2 + v^2 - 2w - 8 = 0 \\ x + z^2 + w - u^2 + 3 = 0 \end{cases}$$

be given.

a) Discuss the solvability for u, v, w in terms of x, y, z near $p_0 = (2, 0, 0, 0, 1, -1)$.

b) Find $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ at p_0 .

a) $\frac{\partial(F, G, H)}{\partial(u, v, w)} = \begin{vmatrix} -3 & 2v & 0 \\ 2u & 2v & -2 \\ -2u & 0 & 1 \end{vmatrix} = \dots = -6 \neq 0$

b) Differentiate w.r.t. x ;

$$\left. \begin{array}{l} 5 - 3 \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} = 0 \\ -2 + 2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} - 2 \frac{\partial w}{\partial x} = 0 \\ 1 + \frac{\partial w}{\partial x} - 2u \frac{\partial u}{\partial x} = 0 \end{array} \right\}$$

So, by the Implicit Function Theorem, u, v and w can be solved in terms of x, y, z near p_0 .

By the Cramer's Rule,

$$\frac{\partial u}{\partial x} \Big|_{p_0} = \left(-\frac{1}{6} \right) \cdot \begin{vmatrix} -5 & 2 & 0 \\ 2 & 2 & -2 \\ -1 & 0 & 1 \end{vmatrix} = \frac{10}{6} = \boxed{\frac{5}{3}}$$

Differentiate now w.r.t. y ;

$$\left. \begin{array}{l} -4 - 3 \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} = 0 \\ 2y + 2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} - 2 \frac{\partial w}{\partial y} = 0 \\ \frac{\partial w}{\partial y} - 2u \frac{\partial u}{\partial y} = 0 \end{array} \right\}$$

By the Cramer's Rule,

$$\frac{\partial u}{\partial y} \Big|_{p_0} = \left(-\frac{1}{6} \right) \begin{vmatrix} -3 & 4 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{vmatrix} = \boxed{0}.$$