MCS 252 2010-2011 Spring
Exercise Set I

1. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) = \dfrac{x^2 \cos y}{z^3}$. Find ∇f.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(x, y) = (x^2 y, xy^3, x^4 y^2)$. Find Df.

3. Let
$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$
Show that f is not differentiable at $(0, 0)$ even though its partial derivatives exist everywhere.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi : \mathbb{R}^3 \to \mathbb{R}$ with $\varphi(r, s, t) = f(rs+rt, rs-rt)$. Find $D\varphi(1, 2, 1)$ if $\dfrac{\partial f}{\partial x}(3, 1) = 4$ and $\dfrac{\partial f}{\partial y}(3, 1) = -5$.

5. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ and $G : \mathbb{R}^2 \to \mathbb{R}^2$ with $F(x, y) = (u(x, y), v(x, y))$ and $G(s, t) = F(s^2+t^2, s^2-t^2)$. Find an expression for the differential matrix of G in terms of the partial derivatives of u and v.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ with
$$\begin{bmatrix} s \\ t \end{bmatrix} = f(u_1, u_2) = \begin{bmatrix} u_1^2 + u_2 \\ u_1 + \sin u_2 \end{bmatrix},$$
and $g : \mathbb{R}^3 \to \mathbb{R}^2$ with
$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = g(x_1, x_2, x_3) = \begin{bmatrix} x_1 x_2 + x_3 \\ x_1 + x_2^2 \end{bmatrix}.$$
Find $D(f \circ g)(x_1, x_2, x_3)$ where $f \circ g : \mathbb{R}^3 \to \mathbb{R}^2$, $(x_1, x_2, x_3) \to (s, t)$.

7. Let $w = w(u_1, u_2, u_3) = u_1^2 + u_2 + u_3$ and
$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = g(x, y, z) = \begin{bmatrix} x + 2yz \\ x^2 + y \\ x + z^2 \end{bmatrix}.$$
Find $\dfrac{\partial w}{\partial x}$, $\dfrac{\partial w}{\partial z}$.

8. Consider
$$T : \begin{cases} u(x,y) = x^3 y + y \sin x \\ v(x,y) = \sin(xy+1) \end{cases}.$$
Near which points (x, y), can we solve x, y in terms of u, v?

9. Compute $\dfrac{\partial x}{\partial u}$ and $\dfrac{\partial y}{\partial v}$ for the transformation given in Problem 8.

10. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by
$$T : \begin{cases} u(x,y,z,t) = x + 2y + 3z + t^2 \\ v(x,y,z,t) = xyt \\ w(x,y,z,t) = 5z^2 + t \\ s(x,y,z,t) = 3y^3 + t + x \end{cases}.$$

(a) Find $DT(1,1,1,0)$.

(b) Can the system of equations be solved for x, y, z as functions of u, v, w near $(1,1,1,0)$?

(c) Find $\dfrac{\partial y}{\partial s}$ and $\dfrac{\partial t}{\partial v}$ at $q=(6,0,5,4)$.

11. Let $T:\mathbb{R}^2\to\mathbb{R}^2$, $T(x,y)=(\ln x+y+1, y^2e^x)$ be given.

   (a) Find DT.

   (b) Let $S:\mathbb{R}^2\to\mathbb{R}$ be a C^1-class function and $h=S\circ T:\mathbb{R}^2\to\mathbb{R}$. Find Dh.

   (c) Does the Inverse Function Theorem apply to T at the point $(1,1)$?

12. Let

$$\begin{cases} x^2u+y^2+3v=0 \\ x+v^3+u^2y=0 \end{cases}$$

be given.

   (a) Are they uniquely solvable for u, v in terms of x and y, near $p_0=(1,1,1,1)$? near $p_1=(0,0,0,1)$?

   (b) Compute $\dfrac{\partial u}{\partial x}$ at $x=1$, $y=1$ if it exists.

   (c) Compute $\dfrac{\partial v}{\partial y}$ at $x=1$, $y=1$ if it exists.

13. Let

$$\begin{cases} x^2+yz+uv=0 \\ x+u^2+v+5w=0 \\ y+z+4u+v^3-2w+1=0 \end{cases}.$$

Discuss the solvability of u, v, w in terms of x, y, z near $p_0=(0,0,0,0,1,1)$. Find $\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial v}{\partial x}$ at p_0.

14. Consider $T:\mathbb{R}^3\to\mathbb{R}^3$ defined by

$$T:\begin{cases} u(x,y,z)=xyz \\ v(x,y,z)=x^2+y-2z \\ w(x,y,z)=3x+4y+z \end{cases}.$$

   (a) Find a domain A for T such that T is locally invertible on A.

   (b) Find DT.

   (c) Find $\dfrac{\partial z}{\partial w}, \dfrac{\partial y}{\partial u}, \dfrac{\partial x}{\partial v}$ at $q=(1,0,8)$.

15. Is it possible to solve

$$\begin{aligned} x^2y+xzu^2-2yv^3&=6 \\ xv+yz^2u^2-3uv&=10 \end{aligned}$$

for $u(x,y,z)$, $v(x,y,z)$ near $(x,y,z)=(1,1,1)$, $(u,v)=(1,-1)$? Compute $\dfrac{\partial v}{\partial y}$.

16. Let $T:\mathbb{R}^4\to\mathbb{R}^2$ be defined by $T(x,y,u,v)=(xv^2+3y+2u^3, x-yv+v^3)$. At which points can we solve $T(x,y,u,v)$ for u, v in terms of x, y? Compute $\dfrac{\partial u}{\partial x}$.

17. Let

$$\begin{aligned} x^2-y^2+2z+u-3v^2&=0 \\ 5y+z-u^2+4w+16&=0 \\ x+2z-w+u^2+4&=0. \end{aligned}$$

Discuss the solvability of the system for u, v, w in terms of x, y, z near $p_0=(0,0,1,1,0,-1)$. Find $\dfrac{\partial u}{\partial x}(p_0)$ and $\dfrac{\partial v}{\partial y}(p_0)$.

18. Show that

$$
\begin{aligned}
u &= x + y \\
v &= x - z \\
w &= y^2 + z^2 - 2yz
\end{aligned}
$$

are functionally dependent.

19. Consider the pair of equations

$$
\begin{aligned}
x^2 + 2ux - y^2 + 3v &= 0 \\
x + 4yu + v^2 + x^2 v &= 0.
\end{aligned}
$$

Find $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$.

20. Let x, y, u, v be related by the equations

$$
\begin{aligned}
xy + x^2 u &= vy^2 \\
3x - 4uy &= x^2 v.
\end{aligned}
$$

Find $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$.